

Suggested Solution to Assignment 3

Exercise 3.1

1. By the method of odd extension or formula (6), we have

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}] e^{-y} dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-\frac{(y+2kt-x)^2}{4kt} + kt-x} - e^{-\frac{(y+2kt+x)^2}{4kt} + kt+x}] dy \\ &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty e^{-p^2} dp - \frac{1}{\sqrt{\pi}} e^{kt+x} \int_{\frac{2kt+x}{\sqrt{4kt}}}^\infty e^{-q^2} dq \\ &= \frac{1}{2} e^{kt-x} [1 - \mathcal{Erf}f(\frac{2kt-x}{\sqrt{4kt}})] - \frac{1}{2} e^{kt+x} [1 - \mathcal{Erf}f(\frac{2kt+x}{\sqrt{4kt}})], \end{aligned}$$

where $\mathcal{Erf}f(x)$ is defined by

$$\mathcal{Erf}f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp. \quad \square$$

2. Let $v(x, t) = u(x, t) - 1$. Then $v(x, t)$ will satisfy

$$v_t = kv_{xx}, \quad v(x, 0) = -1, \quad v(0, t) = 0.$$

Hence,

$$\begin{aligned} v(x, t) &= -\frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}] dy \\ &= -\mathcal{Erf}f(\frac{x}{\sqrt{4kt}}). \\ u(x, t) &= v(x, t) + 1 = 1 - \mathcal{Erf}f(\frac{x}{\sqrt{4kt}}). \quad \square \end{aligned}$$

3. By the method of even reflection, we can translate the original problem for the half-line to the problem for the whole line and then using the formula for the latter to obtain

$$w(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}] \phi(y) dy.$$

For the details, please see your textbook. \square

4. (a) With the rule for differentiation under an integral sign and the property of source function, $v(x, t)$ satisfies

$$v_t = kv_{xx}, \quad v(x, 0) = f(x).$$

(b) By (a), $w(x, t)$ satisfies

$$w_t = kw_{xx}, \quad w(x, 0) = f'(x) - 2f(x).$$

(c) By the definition of f ,

$$\begin{aligned} f'(x) - 2f(x) &= \begin{cases} 1 - 2x, & x > 0; \\ -1 - 2x, & x < 0. \end{cases} \\ f'(-x) - 2f(-x) &= \begin{cases} -1 + 2x, & x > 0; \\ 1 + 2x, & x < 0. \end{cases} \\ &= -[f'(x) - 2f(x)]. \end{aligned}$$

Hence, $f'(x) - 2f(x)$ is an odd function.

- (d) Since $w(x, 0)$ is an odd function, using the conclusion in Exercise 2.4.11, w is an odd function of x .
 (e) By (a), $v(x, t)$ satisfies DE and IC. By (d), $v(x, t)$ satisfies BC. Thus we have proved that $v(x, t)$ satisfies (1) for $x > 0$. Hence, using the assumption for the uniqueness, the solution of (1) is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y) dy,$$

where

$$f(y) = \begin{cases} y, & y > 0; \\ y + 1, & y < 0. \end{cases} \quad \square$$

Exercise 3.2

1. By the method of even extension, we have

$$\begin{aligned} v(x, t) &= \frac{1}{2}[\phi_{\text{even}}(x + ct) + \phi_{\text{even}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{even}}(y) dy \\ &= \begin{cases} \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy, & x \geq ct; \\ \frac{1}{2}[\phi(x + ct) + \phi(-x + ct)] + \frac{1}{2c} [\int_0^{x+ct} \psi(y) dy + \int_0^{-x+ct} \psi(y) dy], & 0 < x < ct. \end{cases} \end{aligned}$$

It is similar for $t < 0$.

2. We can do this problem by even extension, then we obtain the solution to this problem $u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{ext}}(s) ds$, where $\psi_{\text{ext}}(s) = V$ for $a < s < 2a$, $-2a < s < -a$, and zero otherwise. Substitute $t = 0, a/c, 3a/2c, 2a/c, 3a/c$ into this formula and we omit it. \square
3. If the string is fixed at the end $x = 0$, then we have the homogeneous Dirichlet condition $u(0, t) = 0$. Therefore the vibrations $u(x, t)$ of the string for $t > 0$ is given the odd reflection formula with initial data $f(x)$ and $cf'(x)$, that is,

$$u(x, t) = \begin{cases} f(x + ct) & x \geq ct \\ f(x + ct) - f(ct - x) & 0 < x < ct. \end{cases}$$

For details see the formulas (1)-(3) in section 3.2 of the book. \square

5. Using the odd reflection method or formulas(2) and (3), we have

$$u(x, t) = \begin{cases} 1, & x > 2|t|; \\ 0, & x < 2|t|. \end{cases}$$

Hence the singularity is on the lines $x = 2|t|$. \square

6. Since $u_t(0, t) + au_x(0, t) = 0$, we can consider the function $w(x, t)$ defined on the whole line

$$w(x, t) = \begin{cases} u_t(x, t) + au_x(x, t) & x > 0; \\ 0, & x = 0; \\ -u_t(-x, t) - au_x(-x, t), & t < 0. \end{cases}$$

Here, $u_t(0, t) + au_x(0, t) = 0$ enables $w(x, t)$ is continuous and differentiable around $x = 0$. Since $w(x, t)$ is a linear combination of derivatives of $u(x, t)$, it also satisfies the wave equation, that is,

$$w_{tt} = c^2 w_{xx}.$$

By direct calculation,

$$w(x, 0) = \phi(x) = \begin{cases} V, & x > 0; \\ 0, & x = 0; \\ -V, & x < 0. \end{cases}$$

$$w_t(x, 0) = u_{tt}(x, 0) + au_{xt}(x, 0) = c^2 u_{xx}(x, 0) + au_{xt}(x, 0) \\ = c^2 \partial_{xx}^2(0) + a\partial_x(V) = 0.$$

Then the d'Alembert's formula implies

$$w(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] = \begin{cases} V, & x > ct, \\ V/2, & x = ct, \\ 0, & -ct < x < ct, \\ -V/2, & x = -ct, \\ -V, & x < -ct. \end{cases}$$

Let $\varphi(s) = u(x + as, t + s)$, and then $\varphi'(s) = u_t + au_x = w(x + as, t + s)$, $\varphi(-t) = u(x - at, 0) = 0$ and $\varphi(0) = u(x, t)$. Hence,

$$u(x, t) = \int_{-t}^0 w(x + as, t + s) ds.$$

Denote $A = \{(x_1, t_1); 0 \leq t_1 \leq t\} = \{(x_0, t_0); x_0 = ct_0, 0 \leq t_0 \leq t\} \cap \{(x_0, t_0); x - x_0 = a(t - t_0), 0 \leq t_0 \leq t\}$ (i.e. (x_1, t_1) is the point where the line $x_0 = ct_0$ intersects the line $x - x_0 = a(t - t_0)$ when $0 \leq t_0 \leq t$) and $B = \{(x_2, t_2); 0 \leq t_2 \leq t\} = \{(x_0, t_0); x_0 = -ct_0, 0 \leq t_0 \leq t\} \cap \{(x_0, t_0); x - x_0 = a(t - t_0), 0 \leq t_0 \leq t\}$.

Hence, when $x \geq at$, $A = B = \emptyset$ and

$$u(x, t) = \int_{-t}^0 V ds = Vt;$$

when $ct \leq x \leq at$, $t_1 = \frac{at - x}{a - c}$, $t_2 = \frac{at - x}{a + c}$ and

$$u(x, t) = \int_{t_1-t}^0 V ds + \int_{-t}^{t_2-t} -V ds = V \frac{x - ct}{a - c} - V \frac{at - x}{a + c} = V \frac{2ax - (a^2 + c^2)t}{a^2 - c^2};$$

when $0 \leq x \leq ct$, $A = \emptyset$, $t_2 = \frac{at - x}{a + c}$ and

$$u(x, t) = \int_{-t}^{t_2-t} -V ds = -V \frac{at - x}{a + c}. \quad \square$$

Exercise 3.3

- Using the method of reflection and the formula (2) in Section 3.3, we have

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi_{\text{odd}}(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f_{\text{odd}}(y, s) dy ds \\ = \int_0^{\infty} [S(x - y, t) - S(x + y, t)] \phi(y) dy \\ + \int_0^t \int_0^{\infty} [S(x - y, t - s) - S(x + y, t - s)] f(y, s) dy ds,$$

where $f_{\text{odd}}(y, s)$ is the odd extension of $f(y, s)$ w.r.t the variable y , and

$$S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}, \quad t > 0. \quad \square$$

2. Let $V(x, t) = v(x, t) - h(t)$. Then $V(x, t)$ will satisfy

$$\begin{aligned} V_t - kV_{xx} &= f(x, t) - h'(t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty, \\ V(0, t) &= 0, \quad V(x, 0) = \phi(x) - h(0). \end{aligned}$$

Using the result above, we have

$$\begin{aligned} V(x, t) &= \int_0^\infty [S(x-y, t) - S(x+y, t)][\phi(y) - h(0)] dy \\ &\quad + \int_0^t \int_0^\infty [S(x-y, t-s) - S(x+y, t-s)][f(y, s) - h'(t)] dy ds, \\ v(x, t) &= h(t) + \int_0^\infty [S(x-y, t) - S(x+y, t)][\phi(y) - h(0)] dy \\ &\quad + \int_0^t \int_0^\infty [S(x-y, t-s) - S(x+y, t-s)][f(y, s) - h'(t)] dy ds, \end{aligned}$$

where $f_{\text{odd}}(y, s)$ and $S(x, t)$ are shown above. \square

3. Let $W(x, t) = w(x, t) - xh(t)$. Then $W(x, t)$ will satisfy

$$\begin{aligned} W_t - kW_{xx} &= -xh'(t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty, \\ W_x(0, t) &= 0, \quad W(x, 0) = \phi(x) - xh(0). \end{aligned}$$

Using the method of reflection of even functions, we have

$$\begin{aligned} W(x, t) &= \int_{-\infty}^\infty S(x-y, t)\phi_{\text{even}}(y) dy + \int_0^t \int_{-\infty}^\infty S(x-y, t-s)f_{\text{even}}(y, s) dy ds \\ &= \int_0^\infty [S(x-y, t) + S(x+y, t)][\phi(y) - yh(0)] dy \\ &\quad + \int_0^t \int_0^\infty [S(x-y, t-s) + S(x+y, t-s)][-yh'(s)] dy ds, \\ w(x, t) &= W(x, t) + xh(t), \end{aligned}$$

where $f_{\text{even}}(y, s)$ is the even extension of $f(y, s)$ in the variable y , and

$$S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}, \quad t > 0. \quad \square$$

Exercise 3.4

1. By the Theorem 1 in Section 3.4, we have

$$u(x, t) = \frac{1}{2c} \iint_{\Delta} ys \, dy ds = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ys \, dy ds = \frac{xt^3}{6}. \quad \square$$

2. By the Theorem 1 in Section 3.4, we have

$$\begin{aligned} u(x, t) &= \frac{1}{2c} \iint_{\Delta} e^{ay} dy ds = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{ay} dy ds \\ &= \begin{cases} \frac{e^{ax}}{a^2 c^2} \left(\frac{e^{act} + e^{-act}}{2} - 1 \right), & a \neq 0; \\ \frac{1}{2} t^2, & a = 0. \end{cases} \quad \square \end{aligned}$$

3. By the Theorem 1 in Section 3.4, we have

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\sin(x + ct) + \sin(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} (1 + s) ds + \frac{1}{2c} \iint_{\Delta} \cos y dy ds \\ &= \sin x \cos(ct) + (x + 1)t + \frac{1}{c^2} \cos x [1 - \cos(ct)]. \quad \square \end{aligned}$$

4. Let u_1 be the solution of the wave equation

$$u_{tt} = c^2 u_{xx} + f, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0,$$

u_2 be the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = 0,$$

u_3 be the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = 0, \quad u_t(x, 0) = \psi(x).$$

Then $u = u_1 + u_2 + u_3$ is the unique solution for the original problem since the equation and conditions are linear and the uniqueness of the wave equation. Note that u_1 , u_2 , u_3 are terms for f , ϕ and ψ respectively. Hence the solution of the original problem can be written in the sum of three terms, one each for f , ϕ and ψ . \square

5. We write $u(x, t) = \frac{1}{2c} \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s) dy ds$. Then by direct calculation, we have

$$\begin{aligned} u_x &= \frac{1}{2c} \int_0^t [f(x + ct - cs) - f(x - ct + cs)] ds, \quad u_{xx} = \frac{1}{2c} \int_0^t [f'(x + ct - cs) - f'(x - ct + cs)] ds, \\ u_t &= \frac{1}{2} \int_0^t [f(x + ct - cs) + f(x - ct + cs)] ds, \quad u_{tt} = f(x) + \frac{c}{2} \int_0^t [f'(x + ct - cs) - f'(x - ct + cs)] ds. \end{aligned}$$

Hence, we have

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + f \\ u(x, 0) &= \frac{1}{2c} \int_0^0 \int_{x+cs}^{x-cs} f(y, s) dy ds \equiv 0, \\ u_t(x, 0) &= \frac{1}{2} \int_0^0 [f(x - cs) + f(x + cs)] ds \equiv 0. \quad \square \end{aligned}$$

8. For arbitrary C^2 function ψ , $\mathcal{S}\psi = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy$. We have

$$\begin{aligned} [\mathcal{S}\psi]_{tt} &= \frac{c}{2} [\psi'(x + ct) - \psi'(x - ct)] = c^2 [\mathcal{S}\psi]_{xx}. \\ [\mathcal{S}(0)\psi] &= \frac{1}{2c} \int_x^x \psi(y) dy = 0, \quad [\mathcal{S}_t(0)\psi] = \frac{1}{2} [\psi(x) + \psi(x)] = \psi(x). \end{aligned}$$

So we conclude that

$$\mathcal{S}_{tt} - c^2 \mathcal{S}_{xx} = 0, \quad \mathcal{S}(0) = 0, \quad \mathcal{S}_t(0) = I. \quad \square$$

9. According to the definition of $u(x, t)$ and the result above, we have

$$\begin{aligned} u_t &= \mathcal{S}(0)f(t) + \int_0^t \mathcal{S}_t(t-s)f(s)ds = \int_0^t \mathcal{S}_t(t-s)f(s)ds, \\ u_{tt} &= \mathcal{S}_t(0)f(t) + \int_0^t \mathcal{S}_{tt}(t-s)f(s)ds = f(t) + \int_0^t \mathcal{S}_{tt}(t-s)f(s)ds, \\ u_{xx} &= \int_0^t \mathcal{S}_{xx}(t-s)f(s)ds. \end{aligned}$$

So we conclude that

$$u_{tt} - c^2u_{xx} = f, \quad u(x, 0) = \int_0^0 \mathcal{S}(-s)f(s)ds = 0, \quad u_t(0) = \int_0^0 \mathcal{S}_t(-s)f(s)ds = 0 \quad \square$$

12. For $x_0 > ct_0 > 0$, integrate over Δ , where Δ is the region bounded by three lines

$$L_0 = [(x_0 - ct_0, 0), (x_0 + ct_0, 0)], L_1 = [(x_0 + ct_0, 0), (x_0, t_0)], L_2 = [(x_0, t_0), (x_0 - ct_0, 0)]$$

(see figure 6 in Page 76), by Green's theorem, we have

$$\iint_{\Delta} f dx dt = \iint_{\Delta} u_{tt} - c^2u_{xx} dx dt = \int_{L_0+L_1+L_2} -c^2u_x dt - u_t dx$$

On $L_0, dt = 0, u_t(x) = \psi(x), \int_{L_0} -c^2u_x dt - u_t dx = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx.$

On $L_1, x + ct = x_0 + ct_0 \implies dx + cdt = 0, -c^2u_x dt - u_t dx = cu_x dx + cu_t dt = cdu.$

$$\int_{L_1} = c \int_{L_1} du = cu(x_0, t_0) - c\phi(x_0 + ct_0)$$

By the same reasoning, $\int_{L_2} = -c \int_{L_2} du = -c\phi(x_0 - ct_0) + cu(x_0, t_0).$ Summing the three terms, we have for

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi + \frac{1}{2c} \iint_{\Delta} f, \quad \text{if } x > ct > 0. \quad (1)$$

For $x_0 < ct_0$, integrate over Δ' , where Δ' is the reflected region bounded by four lines

$$L_0 = [(ct_0 - x_0, 0), (x_0 + ct_0, 0)], L_1 = [(x_0 + ct_0, 0), (x_0, t_0)],$$

$$L_2 = [(x_0, t_0), (0, t_0 - x_0/c)], L_3 = [(0, t_0 - x_0/c), (ct_0 - x_0, 0)]$$

(see figure 2 in Page 72), by Green's theorem, we have

$$\iint_{\Delta'} f dx dt = \iint_{\Delta'} u_{tt} - c^2u_{xx} dx dt = \int_{L_0+L_1+L_2+L_3} -c^2u_x dt - u_t dx$$

On $L_0, dt = 0, u_t(x) = \psi(x).$ Hence, we have

$$\begin{aligned} \int_{L_0} -c^2u_x dt - u_t dx &= -\int_{ct_0-x_0}^{x_0+ct_0} \psi(x) dx, \\ \int_{L_1} &= c \int_{L_1} du = cu(x_0, t_0) - c\phi(x_0 + ct_0), \\ \int_{L_2} &= -c \int_{L_2} du = -ch(t_0 - x_0/c) + cu(x_0, t_0), \\ \int_{L_3} &= c \int_{L_3} du = c\phi(ct_0 - x_0) - ch(t_0 - x_0/c). \end{aligned}$$

Summing the four terms, we have

$$u(x, t) = \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] - \frac{1}{2c} \int_{ct-x}^{x+ct} \psi + h(t - \frac{x}{c}) + \frac{1}{2c} \iint_{\Delta'} f, \text{ if } 0 < x < ct. \quad (2)$$

13. By the result above, $f \equiv 0$, $\phi(x) \equiv x$, $\psi(x) \equiv 0$ and $h(t) = t^2$ imply that

$$u(x, t) = \begin{cases} \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi + \frac{1}{2c} \iint_{\Delta} f & x \geq ct > 0 \\ \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] - \frac{1}{2c} \int_{ct-x}^{x+ct} \psi + h(t - \frac{x}{c}) + \frac{1}{2c} \iint_{\Delta'} f & 0 < x < ct \end{cases}$$

$$= \begin{cases} x & x \geq ct > 0 \\ x + (t - \frac{x}{c})^2 & 0 < x < ct \end{cases} \quad \square$$

14. Let $v(x, t) = u(x, t) - xk(t)$. Then v satisfies

$$v_{tt} - c^2 v_{xx} = -xk''(t),$$

$$v(x, 0) = -xk(0), \quad v_t(x, 0) = -xk'(0), \quad v_x(0, t) = 0.$$

Then $v_x(0, t) = 0$ enables us to have an even extension. So the solution of v is

$$v(x, t) = \frac{1}{2}[\phi_{\text{even}}(x + ct) + \phi_{\text{even}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{even}} + \frac{1}{2c} \iint_{\Delta} f_{\text{even}},$$

where ϕ_{even} , ψ_{even} and f_{even} are the even extensions of ϕ , ψ and f respectively. Finally, we can have

$$u = \begin{cases} 0 & x \geq ct; \\ -c \int_0^{t-x/c} k(s) ds & x \leq ct. \end{cases} \quad \square$$

Exercise 3.5

1. Since

$$\frac{1}{\sqrt{4\pi}} \int_0^\infty e^{-p^2/4} dp = 1/2,$$

we have

$$\left| \frac{1}{\sqrt{4\pi}} \int_0^\infty e^{-p^2/4} \phi(x + \sqrt{kt}p) dp - \frac{1}{2} \phi(x+) \right| \leq \frac{1}{\sqrt{4\pi}} \int_0^\infty e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp$$

$$+ \frac{1}{\sqrt{4\pi}} \int_{p_0}^\infty e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp + \frac{1}{\sqrt{4\pi}} \int_0^{p_0} e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp$$

For $\forall \epsilon > 0$, choose p_0 large enough such that $\int_{p_0}^\infty e^{-p^2/4} dp$ is small enough and then

$$\frac{1}{\sqrt{4\pi}} \int_{p_0}^\infty e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp \leq C \max|\phi| \int_{p_0}^\infty e^{-p^2/4} dp < \frac{\epsilon}{2};$$

after this, we can choose t is small enough such that

$$|\phi(x + \sqrt{kt}p) - \phi(x+)| < \epsilon$$

and then

$$\frac{1}{\sqrt{4\pi}} \int_0^{p_0} e^{-p^2/4} |\phi(x + \sqrt{kt}p) - \phi(x+)| dp \leq \left(\frac{1}{\sqrt{4\pi}} \int_0^{p_0} e^{-p^2/4} dp \right) \epsilon = \frac{\epsilon}{2}.$$

Hence,

$$\frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \phi(x + \sqrt{kt}p) dp \rightarrow \frac{1}{2} \phi(x+) \quad \text{as } t \searrow 0;$$

similarly we can prove that

$$\frac{1}{\sqrt{4\pi}} \int_0^{-\infty} e^{-p^2/4} \phi(x + \sqrt{kt}p) dp \rightarrow -\frac{1}{2} \phi(x-) \quad \text{as } t \searrow 0. \quad \square$$

2. Since $\phi(x)$ is bounded, by the same argument in Theorem 1, we can show that (1) is an infinitely differentiable solution for $t > 0$. In addition, by Exercise 1,

$$\lim_{t \searrow 0} u(x, t) = \frac{1}{2} [\phi(x+) + \phi(x-)]$$